

Generalized Rothe diagrams for orthogonal roots

arXiv: 2510.25041

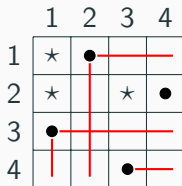
(Joint work with Richard Green)

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Sep. 18, 2026

I. Motivating examples

Traditional Rothe diagrams

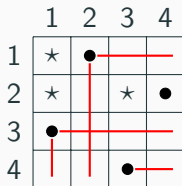


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	1	2	3	4
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2	★		★	●
3	●			
4			●	

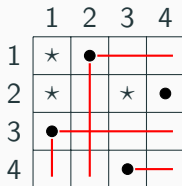
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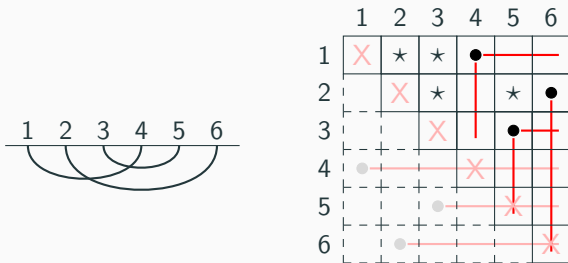
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- The diagrams $D(w)$ can help produce the Schubert polynomials $\mathfrak{S}_w$ (Kohnert, Bergeron, Fomin–Greene–Reiner–Shimozono).

Involution Rothe diagrams

Hamaker–Marberg–Pawlowski define the *involution Rothe diagram* of a fixed-point-free involution $w \in S_{2k}$ to be the set

$$D_{FPF}(w) := \{(i, j) \in [2k] \times [2k] : w(i) > j, w(j) > i, j > i\}.$$



Perfect matching and involution Rothe diagram for $w = (14)(26)(35)$

Involution Rothe diagrams

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$$|D_{FPF}(w)| = \frac{\ell(w) - k}{2} = c(w) + 2 \cdot n(w)$$

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- The diagrams $D_{FPF}(w)$ can help produce the involution Schubert polynomials $\mathfrak{S}_w^{FPF}$ (HMP), which are cohomology class representatives of Sp_{2k} -orbit closures in the flag variety.

A new perspective

Goal: to introduce a root-theoretic framework that can recover collections of objects such as permutations, FPF involutions, their (involution) Rothe diagrams, and more.

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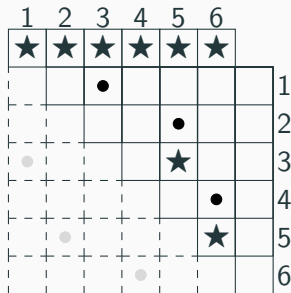
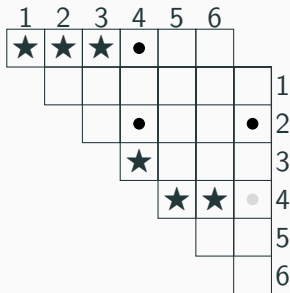
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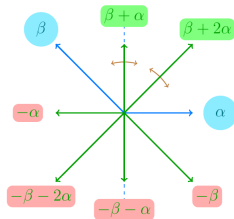
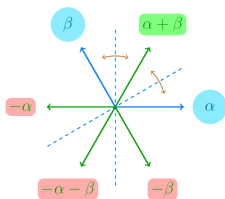
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- the *quasiparabolic* $W(D_4)$ -set (Ω, λ) , with *Poincaré polynomial* $PS_{\Omega}(q) = 1 + q + q^2$;
- the q -Pfaffian $Pf_q = (-q)^0 x_{12}x_{34} + (-q)^1 x_{13}x_{24} + (-q)^2 x_{14}x_{23}$ of Strickland and Jing–Zhang.

Looking ahead: an example in type E



II. Orthogonal roots

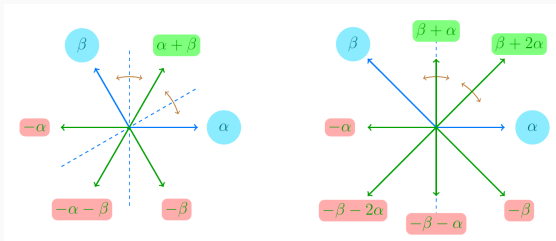
Root systems



Recall that a (*crystallographic*) *root system* in an Euclidean space E is a finite spanning set $\Phi \subseteq E$ of nonzero vectors such that

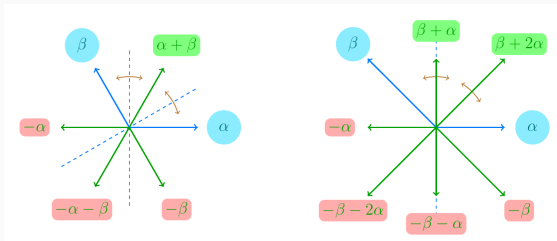
- $\alpha \in \Phi \implies \mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$;
- $\alpha, \beta \in \Phi \implies s_{\alpha}(\beta) := \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha \in \Phi$;
- $\alpha, \beta \in \Phi \implies \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$.

Simple and positive roots



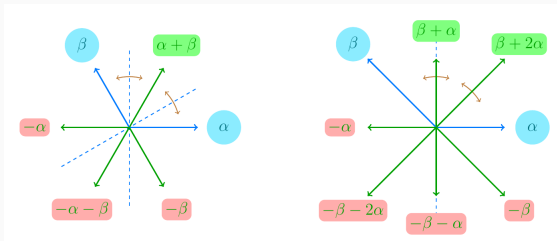
- Every root system $\Phi \subseteq E$ contains a subset Δ such that Δ is a basis of E and $\Phi \subseteq \mathbb{Z}_+\Delta \sqcup \mathbb{Z}_-\Delta$.

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- The elements in Δ , $\mathbb{Z}_+\Delta$, and $\mathbb{Z}_-\Delta$ are called *simple*, *positive*, and *negative* roots (with respect to Δ), respectively.
- We write Φ^+ for the set of positive roots.

The Weyl group

- The *Weyl group* of a root system Φ is the group

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The Weyl group is generated by the set $S = \{s_\alpha : \alpha \in \Delta\}$ of *simple reflections*, and (W, S) is a Coxeter system.

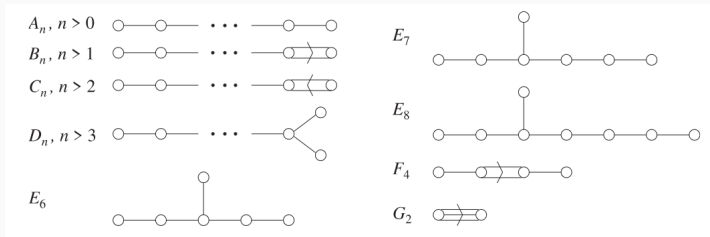
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- Root systems and Weyl groups have a well-known classification, as well as explicit descriptions in many cases.



Dynkin diagrams of irreducible root systems (Wolfram MathWorld)

Explicit descriptions: types A_{n-1} and D_n

In type A_{n-1} , we have:

- $\Phi^+ = \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n\};$
- $\Delta = \{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i \leq n-1\};$
- $W \cong S_n$, with $s_{\varepsilon_i - \varepsilon_j}$ acting as (i, j) on coordinates.

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In type D_n , we have:

- $\Phi^+ = \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n\} \sqcup \{\varepsilon_i + \varepsilon_j : 1 \leq i < j < n\};$
- $\Delta = \{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i \leq n-1\} \sqcup \{\varepsilon_{n-1} + \varepsilon_n\};$
- W acts as signed permutations (with an even number of sign changes), with each $s_{\varepsilon_i + \varepsilon_j}$ sending $\pm \varepsilon_i$ to $\mp \varepsilon_j$ and vice versa.

Explicit descriptions: type E_8

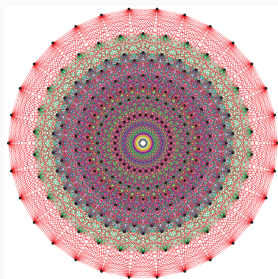
In type E_8 ,

- Φ is the union of $\{\pm 2(\varepsilon_i \pm \varepsilon_j) : 1 \leq i, j \leq 8, i \neq j\}$ and $\{\sum_{i=1}^8 (\pm \varepsilon_i), \text{ signs chosen with an even number of negative signs}\}$.
- $\Delta = \{2\varepsilon_1 + 2\varepsilon_8 - \sum_{i=1}^8 \varepsilon_i\} \sqcup \{\varepsilon_1 + \varepsilon_2\} \sqcup \{\varepsilon_{i-1} - \varepsilon_{i-2} : 3 \leq i \leq 8\}$.

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- W is a group of order 696729600.



A projection of the root system of type E_8 (Stembridge)

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- We are interested in the W -action on the set of k -roots given by

$$w \cdot \{\alpha_1, \dots, \alpha_k\} = \{|w \cdot \alpha_1|, \dots, |w \cdot \alpha_k|\},$$

where each $|\beta|$ denotes the positive root in $\{+\beta, -\beta\}$.

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$$D_U(R) = \{\beta \in U : s_\alpha(\beta) > 0 \text{ for all } \alpha \in R\}.$$

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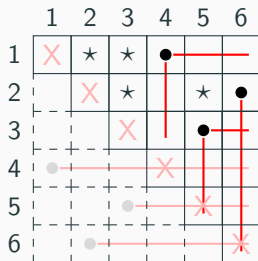
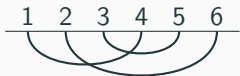
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We will focus on situations where U and Ω_U are W_I -sets for a parabolic subgroup $W_I = \langle I \rangle$ of W ($I \subset S$).

Example: recovery of involution Rothe diagrams

If $n = 2k$ is even, $\Phi = \Phi(D_{2k})$, and $U = \Phi^+$:

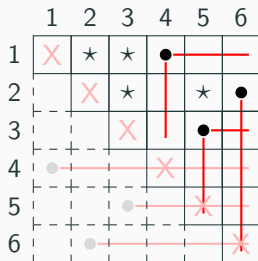
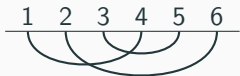


- $\kappa = n$, and there is a natural bijection $FPF(S_{2k}) \rightarrow \Omega_U$ sending each $w_M := (i_1, j_1), (i_2, j_2), \dots, (i_k, j_k) \in FPF(S_{2k})$ to

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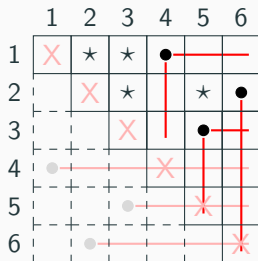
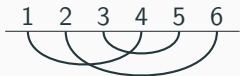
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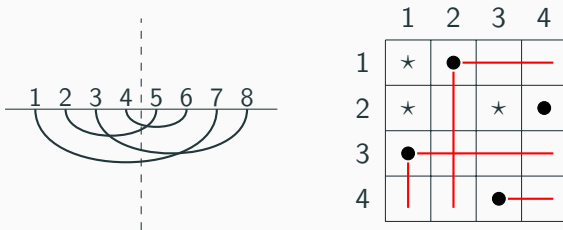
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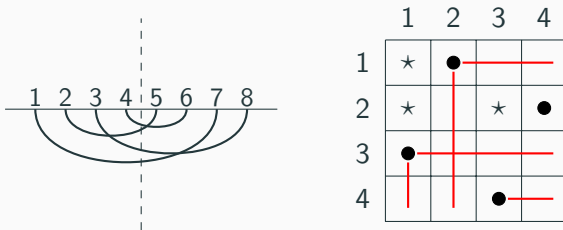


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III. Quasiparabolic structure

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* The defining axioms make it possible to deform the W -action on Ω_U to obtain a module of the Iwahori–Hecke algebra of W .

QP-sets of n -roots

Theorem (Green-X.)

If Φ is a rank- n root system of type D_{2k} , E_7 , or E_8 and $U = \Phi^+$, then $\kappa_U = n$, and (Ω_U, ρ_U) a quasiparabolic W -set.

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- For any $R \in \Omega_{\Phi^+}$, $D_{\Phi^+}(R)$ consists of contributions by the “ D_4 -quadruples” in R , each of which is an “alignment”, a “crossing”, or a “nesting”. In particular, $\lambda(R) = c(R) + 2 \cdot n(R)$.



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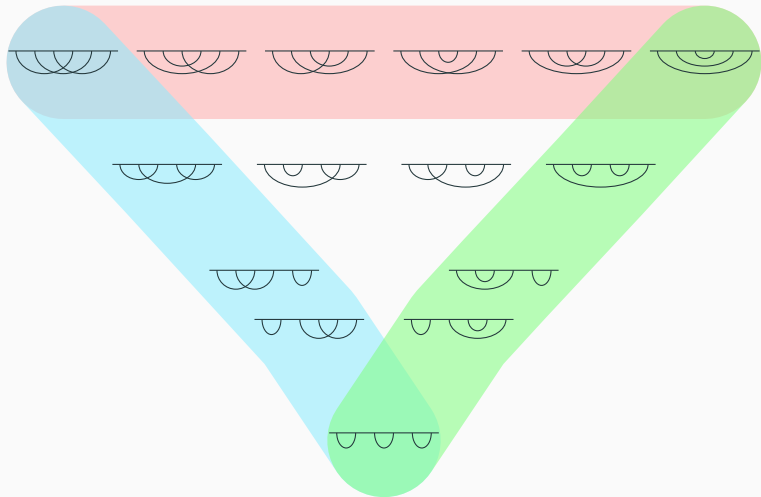
- For any $R \in \Omega_{\Phi^+}$, $D_{\Phi^+}(R)$ consists of contributions by the “ D_4 -quadruples” in R , each of which is an “alignment”, a “crossing”, or a “nesting”. In particular, $\lambda(R) = c(R) + 2 \cdot n(R)$.



- Ω_{Φ^+} gives rise to a *Macdonald representation* $M \subseteq \text{Sym}(V)$ of W , where we have the skein relation “ $C = A + N$ ”.

$$s_{\varepsilon_3 \pm \varepsilon_4} \cdot \begin{array}{c} \text{alignment} \\ \text{(two separate arcs)} \end{array} = \begin{array}{c} \text{crossing} \\ \text{(two overlapping arcs)} \end{array} = \begin{array}{c} \text{alignment} \\ \text{(two separate arcs)} \end{array} + \begin{array}{c} \text{nesting} \\ \text{(one arc inside another)} \end{array}$$

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- The alignment-free n -roots form a QP set for a suitable parabolic subgroup W_I of W .

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- In type D , $T(D_n, p)$ is a QP-triple unless $\lfloor n/2 \rfloor < p < n - 2$.
- In type E , $T(E_n, p)$ is a QP-triple unless $p \in \{3, 4, n - 1\}$.

An example in type A: $T(A_7, 3)$

$$\Gamma = A_7, p = 3 \rightsquigarrow U_3 = \{\varepsilon_i - \varepsilon_j : 1 \leq i \leq 3 < j \leq 8\}, W_I \cong S_3 \times S_5 :$$

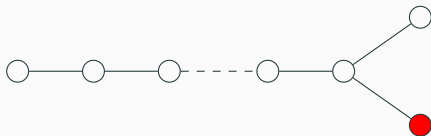


	4	5	6	7	8
1					
2					
3					

$$D_{U_3}(\{\varepsilon_1 - \varepsilon_6, \varepsilon_2 - \varepsilon_7, \varepsilon_3 - \varepsilon_8\})$$

An example in type D : $T(D_{2k}, 2k)$

$$\Gamma = D_{2k}, p = 2k :$$

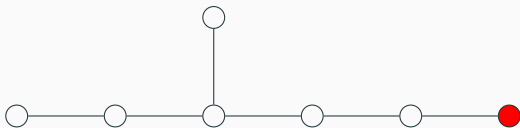


This again recovers our FPF example!

- $U_{2k} = \{\varepsilon_i + \varepsilon_j : 1 \leq i < j \leq 2k\}$, $W_I = W(A_{2k-1}) \cong S_{2k}$.
- The Rothe diagrams are the involution Rothe diagrams.
- The QP order is the restriction of the Bruhat order.

An example in type E : $T(E_7, 7)$

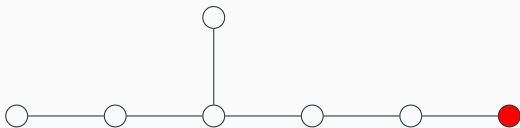
$$\Gamma = E_7, p = 7 \rightsquigarrow W_I \cong W(D_7) :$$



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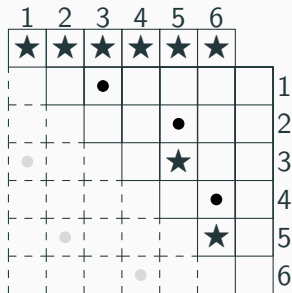
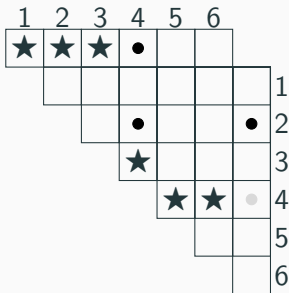
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- $\kappa_{U_7} = 3$, and $\Omega = \Omega_{U_7}$ consists of 45 triples of orthogonal roots, which are in natural bijection with the 45 terms in a cubic invariant polynomial, F , of $L(E_6, \omega_1)$.
- The quantum Hafnian gives a non-recursive way to obtain F , as the “quantum Pfaffian” $QHf|_{q=-1} = \sum_{R \in \Omega} (-1)^{\rho(R)} \prod_{\beta \in R} x_\beta$.

An example in type $E: T(E_7, 7)$



The generalized Rothe diagrams of two elements in Ω_{U_7}

QP-triples of type E

$T(\Gamma, p)$	$ U $	$ \Omega_U $	κ	“degrees”
$T(E_6, 1)$	16	40	2	5, 8
$T(E_6, 2)$	20	30	4	2, 3, 5
$T(E_6, 6)$	16	40	2	5, 8
$T(E_7, 1)$	32	120	4	3, 5, 8
$T(E_7, 2)$	35	30	7	2, 3, 5
$T(E_7, 5)$	35	30	7	2, 3, 5
$T(E_7, 7)$	27	45	3	5, 9
$T(E_8, 1)$	64	240	8	2, 3, 5, 8
$T(E_8, 2)$	64	240	8	2, 3, 5, 8
$T(E_8, 5)$	64	240	8	2, 3, 5, 8
$T(E_8, 6)$	64	240	8	2, 3, 5, 8
$T(E_8, 8)$	56	630	4	5, 9, 14

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- there is a well-defined matroid, M , that can be constructed from any $R \in \Omega$, independent of the choice of R ;
- the sets of the form $T(\Gamma, p)$ correspond to symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ consisting of two simply-laced, equal-rank real Lie algebras, with $U = U_p$ being precisely the noncompact roots of $(\mathfrak{g}, \mathfrak{k})$.

Thank you!